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NAVIER'S SOLUTION FOR LAMINATED PLATE BUCKLING WITH PREBUCKLING IN-PLANE DEFORMATION

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Abstract—This paper presents closed-form buckling solutions for simply supported cross-ply rectangular symmetric laminates with allowance for the effects of prebuckling in-plane deformation and higher-order strain terms (curvature terms), using the Navier solution method. In the formulation, the transverse shear effect is incorporated through the first order shear deformation theory and the incremental total potential energy approach is adopted. Buckling of laminates subject to different combinations of tension and compression along plate edges is investigated. The significant effects of prebuckling in-plane deformation and curvature terms on buckling loads are studied, with various loading conditions, thickness ratios, aspect ratios, degrees of material orthotropy and numbers of layers. Some exact buckling solutions for simplified cases with and without the inclusion of prebuckling in-plane deformation and curvature terms are obtained and compared with results available elsewhere in literature. The investigation concludes that the effect of prebuckling in-plane deformation is of the same order of magnitude as the shear effect with certain kinds of loading conditions and aspect ratios.

1. INTRODUCTION

The growing use of high-performance composite materials in the aerospace industry has stimulated tremendous research interest in structural modelling. The buckling of composite laminates under compression has received considerable attention and numerous research contributions have addressed this matter using both one-dimensional and two-dimensional methods (Whitney and Pagano, 1970; Khdeir, 1988; Khdeir and Librescu, 1988). The effects of prebuckling in-plane deformation and higher-order curvature terms on buckling loads of laminates, however, have not yet been studied.

In open literature, a few studies have examined the buckling analysis of plates with the inclusion of prebuckling in-plane deformation and curvature terms, but only for isotropic materials. The pioneering work of Ziegler (1983) on Mindlin plate buckling with allowance for prebuckling in-plane deformation provided new research opportunities. The same problem was further investigated by Xiang *et al.* (1993) who, in contrast to Ziegler's static-kinematic approach, used an incremental total potential energy approach with the *pb-2* Ritz method for solution. Apart from Ziegler's work on the simply supported case, Xiang *et al.* (1993) presented the first-known buckling results for rectangular isotropic Mindlin plates with arbitrary combinations of boundary conditions.

The primary objectives of this paper are (1) to present a comprehensive investigation on this topic for simply supported symmetric cross-ply laminates; and (2) to determine whether the effect of prebuckling in-plane deformation is of the same order of magnitude as the shear deformation effect on the buckling load factor. To achieve these objectives, Mindlin-type first-order shear deformation plate theory (Yang *et al.*, 1966; Whitney and Pagano, 1970) for laminates has been used and the incremental total potential energy functional has been derived, including terms due to the effects of prebuckling in-plane deformation and curvature.

The governing differential equations for this problem have been derived by considering the stationary condition for the derived energy functional by applying the calculus of variations. Using the Navier solution method, the differential equations have been solved analytically and the critical buckling loads presented in closed-form solutions. The sensitivity of critical buckling loads to the effects of prebuckling in-plane deformation and other factors has been examined.

2. INCREMENTAL POTENTIAL ENERGY

Consider a flat, moderately thick, symmetric cross-ply laminated rectangular plate as shown in Fig. 1. The plate is of thickness t, length a, width b and number of layers l, and is only subjected to in-plane loads $N_x(=N_o)$ and $N_y(=\beta N_o)$.

Let the present configuration of the plate be C_1 and the next configuration be C_2. The incremental total potential energy functional of the plate can be expressed as:

$$\Pi = \frac{1}{2}\int_V \Delta\boldsymbol{\varepsilon}_{\mathrm{L}}^{\mathrm{T}}\bar{\mathbf{Q}}\Delta\boldsymbol{\varepsilon}_{\mathrm{L}}\,\mathrm{d}V + \int_V \boldsymbol{\tau}^{\mathrm{T}}\Delta\boldsymbol{\varepsilon}_{\mathrm{N}}\,\mathrm{d}V - \frac{1}{2}\left(\int_S \Delta N_x u_o\,\mathrm{d}S + \int_S \Delta N_y v_o\,\mathrm{d}S\right), \tag{1}$$

where $\bar{\mathbf{Q}}$ is the material property matrix at C_1; S is the line along the four edges; u_o and v_o are the displacements in the x and y directions at the midplane of the plate when the plate deforms from C_1 to C_2; V is the volume of the plate at C_1; $\Delta\boldsymbol{\varepsilon}$ is the incremental Green–Lagrange strain tensor from C_1 to C_2; ΔN_x and ΔN_y are the load increments in the x and y directions from C_1 to C_2; $\boldsymbol{\tau}$ is the second Piola–Kirchhoff stress tensor at C_1; and the subscripts L and N denote linear and nonlinear components, respectively.

Employing the first-order shear deformation plate theory, the displacement fields of a plate in cartesian coordinates (x, y, z) can be written as:

$$u(x,y,z) = u_o(x,y)+z\theta_x(x,y) \tag{2a}$$

$$v(x,y,z) = v_o(x,y)+z\theta_y(x,y) \tag{2b}$$

$$w(x,y,z) = w_o(x,y), \tag{2c}$$

in which w_o is the displacement from C_1 to C_2 at the midplane of the plate in the z direction; and θ_x and θ_y are the rotations from C_1 to C_2 in y and x directions, respectively (see Fig. 1).

For buckling of a symmetric cross-ply laminated plate, it can be shown that u_o and v_o are uncoupled from w_o, θ_x and θ_y. Thus the displacement fields of the plate can be simplified as

$$u(x,y,z) = z\theta_x(x,y) \tag{3a}$$

$$v(x,y,z) = z\theta_y(x,y) \tag{3b}$$

$$w(x,y,z) = w_o(x,y). \tag{3c}$$

The definitions for $\Delta\boldsymbol{\varepsilon}_{\mathrm{L}}$ and $\Delta\boldsymbol{\varepsilon}_{\mathrm{N}}$ in the Green–Lagrange strain–displacement relation-

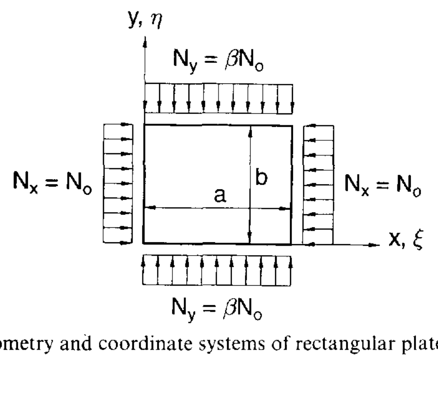

Fig. 1. Geometry and coordinate systems of rectangular plate analyzed.

ship are (Washizu, 1975):

$$\Delta\boldsymbol{\varepsilon}_{\mathrm{L}} = \begin{Bmatrix} \varepsilon_{xx\mathrm{L}} \\ \varepsilon_{yy\mathrm{L}} \\ \gamma_{xy\mathrm{L}} \\ \gamma_{yz\mathrm{L}} \\ \gamma_{zx\mathrm{L}} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y} \\ \dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x} \end{Bmatrix} = \begin{Bmatrix} z\dfrac{\partial \theta_x}{\partial x} \\ z\dfrac{\partial \theta_y}{\partial y} \\ z\left(\dfrac{\partial \theta_x}{\partial y}+\dfrac{\partial \theta_y}{\partial x}\right) \\ \theta_y+\dfrac{\partial w_o}{\partial y} \\ \theta_x+\dfrac{\partial w_o}{\partial x} \end{Bmatrix} \tag{4}$$

$$\Delta\boldsymbol{\varepsilon}_{\mathrm{N}} = \begin{Bmatrix} \varepsilon_{xx\mathrm{N}} \\ \varepsilon_{yy\mathrm{N}} \\ \gamma_{xy\mathrm{N}} \\ \gamma_{yz\mathrm{N}} \\ \gamma_{zx\mathrm{N}} \end{Bmatrix} = \begin{Bmatrix} \dfrac{1}{2}\left[\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial v}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial x}\right)^2\right] \\ \dfrac{1}{2}\left[\left(\dfrac{\partial u}{\partial y}\right)^2+\left(\dfrac{\partial v}{\partial y}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2\right] \\ \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} \\ \dfrac{\partial u}{\partial y}\dfrac{\partial u}{\partial z}+\dfrac{\partial v}{\partial y}\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial z}+\dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial z} \end{Bmatrix}$$

$$= \begin{Bmatrix} \dfrac{1}{2}\left[\underline{z^2\left(\dfrac{\partial \theta_x}{\partial x}\right)^2+z^2\left(\dfrac{\partial \theta_y}{\partial x}\right)^2}+\left(\dfrac{\partial w_o}{\partial x}\right)^2\right] \\ \dfrac{1}{2}\left[\underline{z^2\left(\dfrac{\partial \theta_x}{\partial y}\right)^2+z^2\left(\dfrac{\partial \theta_y}{\partial y}\right)^2}+\left(\dfrac{\partial w_o}{\partial y}\right)^2\right] \\ \underline{z^2\dfrac{\partial \theta_x}{\partial x}\dfrac{\partial \theta_x}{\partial y}+z^2\dfrac{\partial \theta_y}{\partial x}\dfrac{\partial \theta_y}{\partial y}}+\dfrac{\partial w_o}{\partial x}\dfrac{\partial w_o}{\partial y} \\ z\theta_x\dfrac{\partial \theta_x}{\partial y}+z\theta_y\dfrac{\partial \theta_y}{\partial y} \\ z\theta_x\dfrac{\partial \theta_x}{\partial x}+z\theta_y\dfrac{\partial \theta_y}{\partial x} \end{Bmatrix} \tag{5}$$

in which the underlined terms are the higher-order terms in strain $\Delta\boldsymbol{\varepsilon}_{\mathrm{N}}$.

The material property matrix $\bar{\mathbf{Q}}$ for a symmetric cross-ply laminated plate consists of the transformed reduced stiffness matrices, $\bar{\mathbf{Q}}_i$ $(i = 1, 2, \ldots, l)$, for different layers, which can be expressed by

$$\bar{\mathbf{Q}}_i = \begin{bmatrix} \bar{Q}_{11i} & \bar{Q}_{12i} & 0 & 0 & 0 \\ \bar{Q}_{21i} & \bar{Q}_{22i} & 0 & 0 & 0 \\ 0 & 0 & \bar{Q}_{66i} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44i} & 0 \\ 0 & 0 & 0 & 0 & \bar{Q}_{55i} \end{bmatrix} \tag{6}$$

where the elements of $\bar{\mathbf{Q}}_i$ are the so-called reduced stiffnesses, which are given by

$$\bar{Q}_{11i} = Q_{11i}\cos^4\phi_i + Q_{22i}\sin^4\phi_i \tag{7a}$$

$$\bar{Q}_{12i} = \bar{Q}_{21i} = Q_{12i}(\cos^4\phi_i + \sin^4\phi_i) \tag{7b}$$

$$\bar{Q}_{22i} = Q_{11i}\sin^4\phi_i + Q_{22i}\cos^4\phi_i \tag{7c}$$

$$\bar{Q}_{66i} = Q_{66i}(\cos^4\phi_i + \sin^4\phi_i) \tag{7d}$$

$$\bar{Q}_{44i} = Q_{44i}\cos^2\phi_i + Q_{55i}\sin^2\phi_i \tag{7e}$$

$$\bar{Q}_{55i} = Q_{44i}\sin^2\phi_i + Q_{55i}\cos^2\phi_i. \tag{7f}$$

In these equations, $\phi_i(=0^\circ$ or $90^\circ)$ is the fibre orientation for the ith lamina and

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} \tag{8a}$$

$$Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} \tag{8b}$$

$$Q_{21} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} \tag{8c}$$

$$Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} \tag{8d}$$

$$Q_{66} = G_{12} \tag{8e}$$

$$Q_{44} = \kappa_4 G_{23} \tag{8f}$$

$$Q_{55} = \kappa_5 G_{13}, \tag{8g}$$

where E_1 and E_2 are Young's moduli along and transverse to the fibre, respectively; G_{12}, G_{23} and G_{13} are in-plane and transverse shear moduli; ν_{12} and ν_{21} are Poisson's ratios along and transverse to the fibre; and κ_4 and κ_5 are the shear correction factors.

Designating C_1 as the straight configuration (impending state of buckling) and C_2 as the bent configuration (just after buckling), the incremental loads ΔN_x and ΔN_y are equal to zero and the second Piola–Kirchhoff stress tensor at C_1 is given by

$$\begin{aligned}\boldsymbol{\tau}^{\mathrm{T}} &= \{\sigma_x\ \sigma_y\ \tau_{xy}\ \tau_{yz}\ \tau_{zx}\} \\ &= \{-\hat{\sigma}_x\ -\hat{\sigma}_y\ 0\,0\,0\},\end{aligned} \tag{9}$$

where $\hat{\sigma}_x$ and $\hat{\sigma}_y$ consist of the stresses of different laminas, $\hat{\sigma}_{xi}$ and $\hat{\sigma}_{yi}$ $(i = 1, 2, \ldots, l)$, which are related to the in-plane loads N_x and N_y. The procedure for determining $\hat{\sigma}_{xi}$ and $\hat{\sigma}_{yi}$ will be given in due course.

Substituting eqns (4)–(9) into eqn (1) and then integrating with respect to z, the incremental total potential energy functional of a symmetric cross-ply laminated plate is derived as

$$\begin{aligned}\Pi = \frac{1}{2}\int_A \Bigg\{ & D_{11}\left(\frac{\partial\theta_x}{\partial x}\right)^2 + D_{22}\left(\frac{\partial\theta_y}{\partial y}\right)^2 + D_{66}\left(\frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\right)^2 \\ & + 2D_{12}\frac{\partial\theta_x}{\partial x}\frac{\partial\theta_y}{\partial y} + A_{44}\left(\theta_y + \frac{\partial w_o}{\partial y}\right)^2 + A_{55}\left(\theta_x + \frac{\partial w_o}{\partial x}\right)^2 \\ & - N_x\left(\frac{\partial w_o}{\partial x}\right)^2 - N_y\left(\frac{\partial w_o}{\partial y}\right)^2 \underline{- F_x\left[\left(\frac{\partial\theta_x}{\partial x}\right)^2 + \left(\frac{\partial\theta_y}{\partial x}\right)^2\right]} \\ & \underline{- F_y\left[\left(\frac{\partial\theta_x}{\partial y}\right)^2 + \left(\frac{\partial\theta_y}{\partial y}\right)^2\right]}\Bigg\}\mathrm{d}A,\end{aligned} \tag{10}$$

in which the underlined terms are the so-called curvature terms which come from the higher-order strain terms in eqn (5). The parameters A_{jk}, D_{jk}, F_x and F_y are determined by

$$A_{jk} = \sum_{i=1}^{l} \bar{Q}_{jki}(t_i - t_{i-1}) \quad j,k = 4,5 \tag{11a}$$

$$D_{jk} = \frac{1}{3}\sum_{i=1}^{l} \bar{Q}_{jki}(t_i^3 - t_{i-1}^3) \quad j,k = 1,2,6 \tag{11b}$$

$$F_x = \frac{1}{3}\sum_{i=1}^{l} \hat{\sigma}_{xi}(t_i^3 - t_{i-1}^3) \tag{11c}$$

$$F_y = \frac{1}{3}\sum_{i=1}^{l} \hat{\sigma}_{yi}(t_i^3 - t_{i-1}^3). \tag{11d}$$

A is the plate area at the impending state of buckling C_1, given by

$$A = (a\Lambda_x)(b\Lambda_y), \tag{12}$$

where Λ_x and Λ_y are the pre-buckling in-plane deformation factors in the x and y directions, respectively.

When the plate is at the C_1 configuration (impending buckling), the in-plane strains $\hat{\varepsilon}_x$ and $\hat{\varepsilon}_y$ can be obtained by:

$$\begin{Bmatrix} \hat{\varepsilon}_x \\ \hat{\varepsilon}_y \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \begin{Bmatrix} N_x \\ N_y \end{Bmatrix} \tag{13a}$$

$$A_{jk} = \sum_{i=1}^{l} \bar{Q}_{jki}(t_i - t_{i-1}) \quad j,k = 1,2. \tag{13b}$$

The above equation implies that the strains $\hat{\varepsilon}_x$ and $\hat{\varepsilon}_y$ are identical in different laminas under in-plane loads N_x and N_y, respectively. Thus, the prebuckling in-plane deformation factors Λ_x and Λ_y can be determined by

$$\Lambda_x = 1 - \gamma\hat{\varepsilon}_x \tag{14a}$$

$$\Lambda_y = 1 - \gamma\hat{\varepsilon}_y, \tag{14b}$$

in which γ is a scalar indicator. If $\gamma = 1$, the effect of prebuckling in-plane deformation is considered; while if $\gamma = 0$, this effect is ignored. Furthermore, the stresses $\hat{\sigma}_{xi}$ and $\hat{\sigma}_{yi}$ of the ith lamina are given by

$$\begin{Bmatrix} \hat{\sigma}_{xi} \\ \hat{\sigma}_{yi} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11i} & \bar{Q}_{12i} \\ \bar{Q}_{21i} & \bar{Q}_{22i} \end{bmatrix} \begin{Bmatrix} \hat{\varepsilon}_x \\ \hat{\varepsilon}_y \end{Bmatrix}. \tag{15}$$

For generality and convenience, the cartesian coordinates (x, y) are non-dimensionalized as follows:

$$\xi = \frac{x}{a\Lambda_x} \tag{16a}$$

$$\eta = \frac{y}{b\Lambda_y}. \tag{16b}$$

Substituting eqn (16) into eqn (10) leads to

$$\begin{aligned}\Pi = \frac{1}{2}\int_0^1\int_0^1 \Bigg\{ & D_{11}\left(\frac{1}{a\Lambda_x}\frac{\partial\theta_x}{\partial\xi}\right)^2 + D_{22}\left(\frac{1}{b\Lambda_y}\frac{\partial\theta_y}{\partial\eta}\right)^2 \\ & + D_{66}\left(\frac{1}{b\Lambda_y}\frac{\partial\theta_x}{\partial\eta} + \frac{1}{a\Lambda_x}\frac{\partial\theta_y}{\partial\xi}\right)^2 + 2D_{12}\frac{1}{ab\Lambda_x\Lambda_y}\frac{\partial\theta_x}{\partial\xi}\frac{\partial\theta_y}{\partial\eta} \\ & + A_{44}\left(\theta_y + \frac{1}{b\Lambda_y}\frac{\partial w_o}{\partial\eta}\right)^2 + A_{55}\left(\theta_x + \frac{1}{a\Lambda_x}\frac{\partial w_o}{\partial\xi}\right)^2 \\ & - N_o\left[\left(\frac{1}{a\Lambda_x}\frac{\partial w_o}{\partial\xi}\right)^2 + \beta\left(\frac{1}{b\Lambda_y}\frac{\partial w_o}{\partial\eta}\right)^2\right] \\ & \underline{- \mu F_x\left[\left(\frac{1}{a\Lambda_x}\frac{\partial\theta_x}{\partial\xi}\right)^2 + \left(\frac{1}{a\Lambda_x}\frac{\partial\theta_y}{\partial\xi}\right)^2\right]} \\ & \underline{- \mu F_y\left[\left(\frac{1}{b\Lambda_y}\frac{\partial\theta_x}{\partial\eta}\right)^2 + \left(\frac{1}{b\Lambda_y}\frac{\partial\theta_y}{\partial\eta}\right)^2\right]}\Bigg\} ab\Lambda_x\Lambda_y \, \mathrm{d}\xi \, \mathrm{d}\eta. \end{aligned} \tag{17}$$

A scalar indicator μ is introduced in the above equation. If $\mu = 1$, the higher-order terms (curvature terms) are included in the energy functional. If $\mu = 0$, the curvature terms are neglected.

3. GOVERNING DIFFERENTIAL EQUATIONS

Taking the stationary conditions of eqn (17) with respect to w_o, θ_x and θ_y using the Euler–Lagrange equation (Schechter, 1967) yields the following governing differential equations for buckling of symmetric cross-ply laminated plates with allowance for pre-buckling in-plane deformation and curvature terms:

$$\begin{aligned}\frac{\partial}{\partial\xi}\left[\frac{1}{a^2\Lambda_x^2}(D_{11}-\mu F_x)\frac{\partial\theta_x}{\partial\xi} + \frac{D_{12}}{ab\Lambda_x\Lambda_y}\frac{\partial\theta_y}{\partial\eta}\right] + \frac{\partial}{\partial\eta}\left[\frac{1}{b^2\Lambda_y^2}(D_{66}-\mu F_y)\frac{\partial\theta_x}{\partial\eta} + \frac{D_{66}}{ab\Lambda_x\Lambda_y}\frac{\partial\theta_y}{\partial\xi}\right] \\ - A_{55}\left(\theta_x + \frac{1}{a\Lambda_x}\frac{\partial w_o}{\partial\xi}\right) = 0\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{\partial}{\partial\xi}\left[\frac{1}{a^2\Lambda_x^2}(D_{66}-\mu F_x)\frac{\partial\theta_y}{\partial\xi} + \frac{D_{66}}{ab\Lambda_x\Lambda_y}\frac{\partial\theta_x}{\partial\eta}\right] + \frac{\partial}{\partial\eta}\left[\frac{1}{b^2\Lambda_y^2}(D_{22}-\mu F_y)\frac{\partial\theta_y}{\partial\eta} + \frac{D_{12}}{ab\Lambda_x\Lambda_y}\frac{\partial\theta_x}{\partial\xi}\right] \\ - A_{44}\left(\theta_y + \frac{1}{b\Lambda_y}\frac{\partial w_o}{\partial\eta}\right) = 0\end{aligned} \tag{19}$$

$$\frac{A_{55}}{a\Lambda_x}\frac{\partial}{\partial\xi}\left(\theta_x+\frac{1}{a\Lambda_x}\frac{\partial w_o}{\partial\xi}\right)+\frac{A_{44}}{b\Lambda_y}\frac{\partial}{\partial\eta}\left(\theta_y+\frac{1}{b\Lambda_y}\frac{\partial w_o}{\partial\eta}\right)$$

$$-N_o\left[\frac{1}{a^2\Lambda_x^2}\frac{\partial}{\partial\xi}\left(\frac{\partial w_o}{\partial\xi}\right)+\frac{\beta}{b^2\Lambda_y^2}\frac{\partial}{\partial\eta}\left(\frac{\partial w_o}{\partial\eta}\right)\right]=0. \quad (20)$$

For simply supported symmetric cross-ply laminated rectangular plates, the following boundary conditions need to be satisfied.

(a) For the edges parallel to x-axis:

$$w_o = M_y = \theta_x = 0. \quad (21)$$

(b) For the edges parallel to y-axis:

$$w_o = M_x = \theta_y = 0, \quad (22)$$

in which M_x and M_y are bending moments in y and x directions which are given by

$$M_x = \frac{D_{11}-F_x}{a\Lambda_x}\frac{\partial\theta_x}{\partial\xi}+\frac{D_{12}}{b\Lambda_y}\frac{\partial\theta_y}{\partial\eta} \quad (23a)$$

$$M_y = \frac{D_{22}-F_y}{b\Lambda_y}\frac{\partial\theta_y}{\partial\eta}+\frac{D_{12}}{a\Lambda_x}\frac{\partial\theta_x}{\partial\xi}. \quad (23b)$$

4. SOLUTION METHOD

Let the origin of coordinates (ξ,η) be at the bottom left corner of the plate. The trigonometric functions

$$\theta_x(\xi,\eta) = A_{mn}\cos(m\pi\xi)\sin(n\pi\eta) \quad (24a)$$

$$\theta_y(\xi,\eta) = B_{mn}\sin(m\pi\xi)\cos(n\pi\eta) \quad (24b)$$

$$w_o(\xi,\eta) = C_{mn}\sin(m\pi\xi)\sin(n\pi\eta) \quad (24c)$$

satisfy all the boundary conditions of a simply supported symmetric cross-ply laminated rectangular plate [see eqns (21)–(23)]. In eqn (24), A_{mn}, B_{mn} and C_{mn} are constants and m, $n = 1, 2, \ldots, \infty$.

Substituting eqn (24) into the differential equations [eqns (18)–(20)] the governing eigenvalue equation, which determines the buckling load of the simply supported rectangular plate, is derived as

$$\begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6-N_op_7 \end{bmatrix}\begin{Bmatrix} A_{mn} \\ B_{mn} \\ C_{mn} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad (25)$$

in which

$$p_1 = \pi^2\left(\frac{D_{11}m^2}{a^2\Lambda_x^2}+\frac{D_{66}n^2}{b^2\Lambda_y^2}\right)+A_{55}-\mu\pi^2\left(F_x\frac{m^2}{a^2\Lambda_x^2}+F_y\frac{n^2}{b^2\Lambda_y^2}\right) \quad (26a)$$

$$p_2 = (D_{12}+D_{66})\frac{mn\pi^2}{ab\Lambda_x\Lambda_y} \quad (26b)$$

$$p_3 = \frac{A_{55} m\pi}{a\Lambda_x} \tag{26c}$$

$$p_4 = \pi^2\left(\frac{D_{66}m^2}{a^2\Lambda_x^2} + \frac{D_{22}n^2}{b^2\Lambda_y^2}\right) + A_{44} - \mu\pi^2\left(F_x\frac{m^2}{a^2\Lambda_x^2} + F_y\frac{n^2}{b^2\Lambda_y^2}\right) \tag{26d}$$

$$p_5 = \frac{A_{44} n\pi}{b\Lambda_y} \tag{26e}$$

$$p_6 = \pi^2\left(\frac{A_{55}m^2}{a^2\Lambda_x^2} + \frac{A_{44}n^2}{b^2\Lambda_y^2}\right) \tag{26f}$$

$$p_7 = \pi^2\left(\frac{m^2}{a^2\Lambda_x^2} + \frac{\beta n^2}{b^2\Lambda_y^2}\right). \tag{26g}$$

For nontrivial solution of the buckling load N_o, the determinant of the matrix in eqn (25) must be equal to zero. It is seen from eqns (13) and (14) that the pre-buckling in-plane deformation factors Λ_x and Λ_y are functions of in-plane load N_o. Consequently, the terms p_i, $i = 1, 2, \ldots, 7$, in eqn (26) are all functions of N_o and the determinant of the matrix in eqn (25) is a high-order function of N_o. In view of the difficulty involved in a continued analytical development, a numerical procedure is implemented to obtain closed-form solutions for the buckling load N_o.

The influence of load N_o in p_i, $i = 1$–7, is revealed through the prebuckling in-plane deformation terms Λ_x and Λ_y, and the so-called curvature terms (when $\mu = 1$). Viewing eqn (25), the buckling load N_o can be expressed as

$$N_o = \frac{p_3^2 p_4 - 2p_2 p_3 p_5 + p_1 p_5^2 + p_2^2 p_6 - p_1 p_4 p_6}{p_7(p_2^2 - p_1 p_4)}. \tag{27}$$

As p_i, $i = 1$–7, are functions of N_o, the buckling load N_o cannot be determined directly from eqn (27). Where the effects of prebuckling in-plane deformation and curvature terms are not considered, p_i are not functions of N_o and the buckling load is given directly.

A general iterative procedure for obtaining the buckling load N_o is as follows:

(1) The effects of pre-buckling in-plane deformation and curvature terms are ignored, thus $\Lambda_x = \Lambda_y = 1$ and $\mu = 0$; p_i, $i = 1$–7 are calculated.
(2) Substitute p_i, $i = 1$–7, into eqn (27) to obtain N_o.
(3) Substitute the calculated N_o into eqns (11)–(15) to calculate Λ_x, Λ_y, F_x and F_y, and set $\mu = 1$; substitute the calculated Λ_x, Λ_y, F_x and F_y into eqn (26) to obtain a new set of p_i, $i = 1$–7.
(4) Repeat steps (2) and (3) until the relative error in the values N_o between two successive solutions is within a given tolerance. In this study, the tolerance has been set to be 1/100 000.

Note that since $m, n = 1, 2, \ldots, \infty$, there is an infinite number of buckling loads N_o. The critical buckling load is the minimum positive real solution with respect to m and n.

5. NUMERICAL STUDIES

Several examples are presented in this section to examine the effects of prebuckling in-plane deformation and curvature terms on the critical buckling factor, $\lambda = N_o b^2/(E_2 t^3)$, for simply supported symmetric cross-ply laminates. Variations in respect to aspect ratio a/b, thickness ratio t/b, in-plane load ratio N_y/N_x and degree of orthotropy E_1/E_2 have been considered.

In each example, the plate has been assumed to be composed of an odd number of layers which have the same geometric and material properties, and lie in the order $(0°/90°)_{sym}$. The material properties have been taken to be $E_1/E_2 = 3$, $G_{12} = G_{13} = 0.6\,E_2$, $G_{23} = 0.5\,E_2$ and $\nu_{12} = 0.25$. Shear correction factors $\kappa_4 = \kappa_5 = 5/6$ have been used unless stated otherwise.

Four different cases have been considered in the numerical study: (1) without the effects of pre-buckling in-plane deformation and curvature terms; (2) with the effect of prebuckling in-plane deformation only; (3) with the effect of curvature terms only; and (4) with the effects of both prebuckling in-plane deformation and curvature terms. Note that Case 1 is the conventional consideration of thick plate buckling, which forms the basis of comparison for the other three cases.

5.1. *Comparison studies*

The correctness of the Navier procedure used in this paper was first verified by comparing results with solutions available in open literature. Table 1 presents the buckling results obtained by the authors, Khdeir and Librescu (1988) and Khdeir (1988) for simply supported symmetric cross-ply laminated square plates ($t/b = 0.1$) subjected to uniaxial compression. The results are presented for square plates having varying degrees of orthotropy and numbers of layers without consideration of the effects of prebuckling in-plane deformation and curvature terms. The buckling factors obtained are almost identical to those of Khdeir and Librescu (1988) who employed the first-order shear deformation plate theory in their study. Further, the results generated using the first-order shear deformation plate theory are only slightly different from those derived using the higher-order shear deformation plate theory (Khdeir, 1988). Therefore, the first-order shear deformation plate theory can predict buckling factors with reasonably high accuracy for moderately thick rectangular plates.

For plate buckling with consideration of the effects of prebuckling in-plane deformation and curvature terms, no results are available for laminates. Table 2 gives the buckling factors obtained by the authors and Xiang *et al.* (1993) for simply supported isotropic rectangular plates with consideration of the effects of prebuckling in-plane deformation and curvature terms. Very close agreement has been achieved between results obtained by the authors and Xiang *et al.* (1993). These comparison studies confirm the validity and accuracy of the proposed solution method.

5.2. *Parametric studies*

The effects of prebuckling in-plane deformation and curvature terms on the buckling factors have been first investigated with respect to aspect ratio. Figures 2–7 present the

Table 1. Comparison of buckling factors, $N_o b^2/E_2 t^3$, for simply supported symmetric cross-ply laminated square plates ($t/b = 0.1$) subject to uniaxial compression ($N_x = N_o$, $N_y = 0$) without consideration of the effects of prebuckling in-plane deformation and curvature terms

Theories	Number of layers	E_1/E_2		
		20	30	40
Present		14.9846	19.0265	22.3151
FSDT†	3	14.985	19.027	22.315
HSDPT‡		14.890	18.878	22.121
Present		15.7361	20.4847	24.5465
FSDT†	5	15.736	20.485	24.547
HSDPT‡		15.783	20.578	24.676
Present		16.0682	21.1171	25.4945
FSDT†	9	16.068	21.117	25.495
HSDPT‡		16.101	21.178	25.585

†Results reported by Khdeir and Librescu (1988) obtained by using the first-order shear deformation theory.

‡Results reported by Khdeir (1988) obtained by using the higher-order shear deformation plate theory.

Table 2. Comparison of buckling factors, $N_o b^2/\pi^2 D_{11}$, for simply supported isotropic rectangular plates subject to different combinations of in-plane loads with consideration of the effects of prebuckling in-plane deformation and curvature terms

Sources	Load conditions	$a/b = 1.0$			$a/b = 2.0$		
		t/b			t/b		
		0.05	0.10	0.15	0.05	0.10	0.15
Present	$N_x = N_o, N_y = 0$	3.90734	3.65692	3.31097	3.90734	3.65692	3.31097
Xiang *et al.* (1993)		3.9074	3.6569	3.3110	3.9074	3.6569	3.3110
Present	$N_x = N_y = N_o$	1.97611	1.90717	1.80083	1.24063	1.21321	1.16967
Xiang *et al.* (1993)		1.9761	1.9072	1.8008	1.2406	1.2132	1.1697
Present	$N_x = N_o, N_y = -N_o$	7.81926	6.72074	5.56055	7.61382	6.17773	4.95782
Xiang *et al.* (1993)		7.8193	6.7207	5.5606	7.6138	6.1778	4.9578

The properties of the isotropic material are: $E_1/E_2 = 1$, $G_{12} = G_{13} = G_{23} = E_2/2(1+\nu_{12})$, $\nu_{12} = \nu_{21} = 0.3$ and $\kappa_4 = \kappa_5 = 20(1+\nu_{12})/(24+25\nu_{12}+\nu_{12}^2)$.

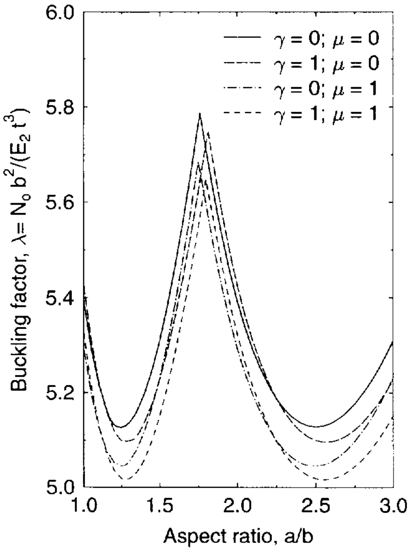

Fig. 2. Buckling factor, λ, vs aspect ratio, a/b, for simply supported symmetric cross-ply rectangular laminates ($E_1/E_2 = 3$, $t/b = 0.1$, three-ply) subject to uniaxial compression ($N_x = N_o$, $N_y = 0$) with different combinations of the effects of prebuckling in-plane deformation and curvature terms.

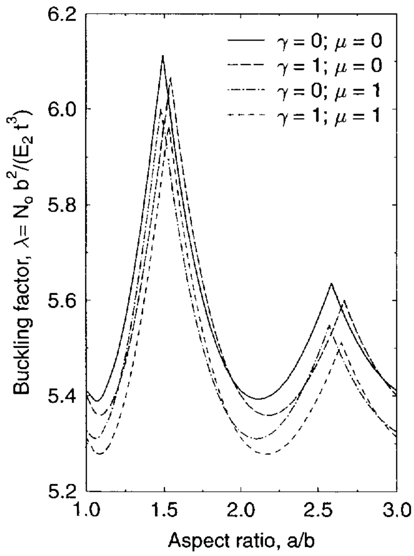

Fig. 3. Buckling factor, λ, vs aspect ratio, a/b, for simply supported symmetric cross-ply rectangular laminates ($E_1/E_2 = 3$, $t/b = 0.1$, nine-ply) subject to uniaxial compression ($N_x = N_o$, $N_y = 0$) with different combinations of the effects of prebuckling in-plane deformation and curvature terms.

buckling factor vs aspect ratio for three- and nine-ply plates with thickness ratio $t/b = 0.1$ and degree of orthotropy $E_1/E_2 = 3$ under different combinations of load conditions.

Buckling factors are plotted against aspect ratio for plates under uniaxial compression ($N_x = N_o$ and $N_y = 0$) in Figs 2 and 3. If only the effect of prebuckling in-plane deformation (Case 2) is considered, the buckling factors may be either greater or less than those in Case 1 depending on the values of the aspect ratios. On the other hand, if only the effect of curvature terms (Case 3) is included, the buckling factors are always lower than those in Case 1. When both effects (Case 4) are considered, the buckling factors are a combination of Cases 2 and 3. Note that the buckling mode shifting points occur at similar aspect ratios in Cases 1 and 3 and in Cases 2 and 4. This may be explained by the fact that the plate dimensions are the same in Cases 1 and 3 and in Cases 2 and 4. Comparing Figs 2 and 3, the influence of the number of plies on the effects of prebuckling in-plane deformation and

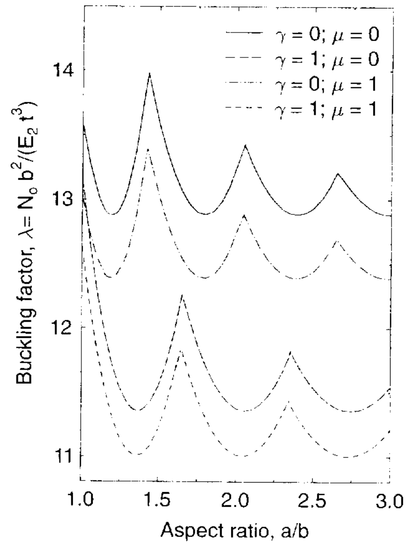

Fig. 4. Buckling factor, λ, vs aspect ratio, a/b, for simply supported symmetric cross-ply rectangular laminates ($E_1/E_2 = 3$, $t/b = 0.1$, three-ply) subject to compression along the x-axis and tension along the y-axis ($N_x = N_o$, $N_y = -N_o$) with different combinations of the effects of prebuckling in-plane deformation and curvature terms.

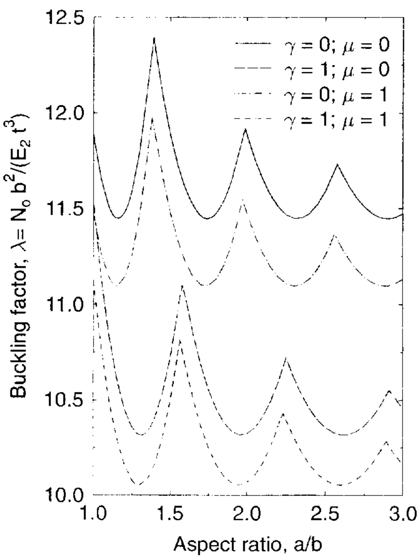

Fig. 5. Buckling factor, λ, vs aspect ratio, a/b, for simply supported symmetric cross-ply rectangular laminates ($E_1/E_2 = 3$, $t/b = 0.1$, nine-ply) subject to compression along the x-axis and tension along the y-axis ($N_x = N_o$, $N_y = -N_o$) with different combinations of the effects of prebuckling in-plane deformation and curvature terms.

curvature terms is insignificant. The maximum difference in buckling factors between Cases 1 and 4 is about 4% for three-ply plate at $a/b \approx 1.7$ and for nine-ply plate at $a/b \approx 1.4$.

Figures 4 and 5 show the buckling factor against aspect ratio for plates subject to compression along the x-axis and tension along the y-axis ($N_x = N_o$ and $N_y = -N_o$). The effects of prebuckling in-plane deformation and curvature terms (Cases 2–4) on buckling factors are much greater than in the previously discussed loading condition. Unlike the previous loading condition, the inclusion of prebuckling in-plane deformation (Case 2) decreases the buckling factor for all aspect ratios considered. The mode shift points in Cases 1 and 3 and in Cases 2 and 4 again occur at almost the same aspect ratios. The mode shift points in Cases 2 and 4 move away significantly from those in Cases 1 and 3 which makes the effects of prebuckling in-plane deformation and curvature terms on the buckling

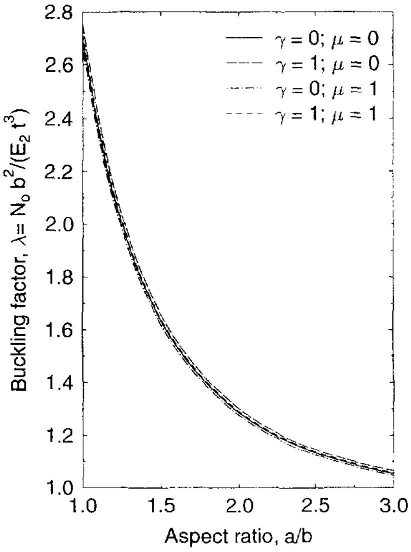

Fig. 6. Buckling factor, λ, vs aspect ratio, a/b, for simply supported symmetric cross-ply rectangular laminates ($E_1/E_2 = 3$, $t/b = 0.1$, three-ply) subject to biaxial compression ($N_x = N_y = N_o$) with different combinations of the effects of prebuckling in-plane deformation and curvature terms.

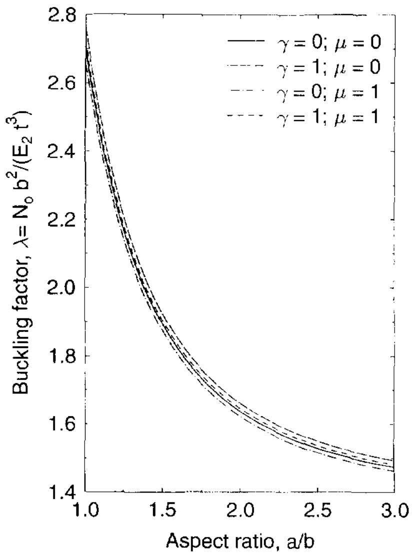

Fig. 7. Buckling factor, λ, vs aspect ratio, a/b, for simply supported symmetric cross-ply rectangular laminates ($E_1/E_2 = 3$, $t/b = 0.1$, nine-ply) subject to biaxial compression ($N_x = N_y = N_o$) with different combinations of the effects of prebuckling in-plane deformation and curvature terms.

factors much more pronounced at some aspect ratios. The buckling factor decreases more significantly in Case 2 than in Case 3 under this loading condition. The influence of the number of plies on the effects of prebuckling in-plane deformation and curvature terms is not obvious. The maximum difference between buckling factors for Cases 1 and 4 is about 20% for three-ply plate at $a/b \approx 1.4$ and about 18% for nine-ply plate at around the same location.

Figures 6 and 7 provide buckling information for plates under biaxial compression ($N_x = N_y = N_o$). The effects of prebuckling in-plane deformation and curvature terms (Cases 2–4) on the buckling factors are very small by comparison with Case 1 and may be neglected. The buckling factors in Cases 2 and 4 are always higher than in Case 1, while the buckling factors in Case 3 are lower than in Case 1. The effects of prebuckling in-plane

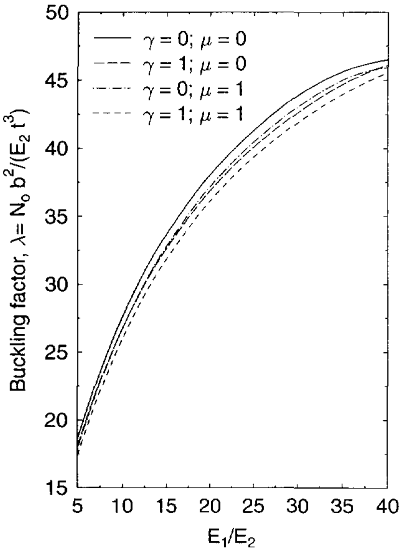

Fig. 8. Buckling factor, λ, vs degree of orthotropy, E_1/E_2, for simply supported symmetric cross-ply square laminates ($a/b = 1$, $t/b = 0.1$, three-ply) subject to compression along the x-axis and tension along the y-axis ($N_x = N_o$, $N_y = -N_o$) with different combinations of the effects of pre-buckling in-plane deformation and curvature terms.

deformation and curvature terms on the buckling factor are slightly greater in nine-ply plate than in three-ply plate.

As discussed, the effects of prebuckling in-plane deformation and curvature terms on the buckling factors are most significant in plates subjected to compression along the x-axis and tension along the y-axis. The following analyses are devoted to this loading condition only.

Figure 8 presents the relationship between the buckling factor λ and the degree of orthotropy E_1/E_2 for square plates ($t/b = 0.1$) subject to compression along the x-axis and tension along the y-axis. The variation in buckling factors between Case 1 and Case 2, 3 or 4 changes marginally with increasing E_1/E_2 ratio.

Figure 9 investigates the buckling behaviour of the same plates with varying plate thickness ratio t/b. When a plate is thin ($t/b = 0.001$), the effects of prebuckling in-plane deformation and curvature terms (Cases 2–4) on the buckling factors are insignificant, as expected. These effects become more pronounced as the plate thickness ratio t/b increases, with the buckling factors decreasing in all four cases due to the effect of transverse shear deformation. When the thickness ratio of a plate is 0.1, the buckling factor is about 18% lower than the thin plate solution ($t/b = 0.001$) because of the effect of transverse shear deformation (Case 1), 20.7% lower with inclusion of the effect of prebuckling in-plane deformation (Case 2), 21.5% lower with inclusion of the effect of curvature terms (Case 3) and 23.9% lower with inclusion of both effects (Case 4). Observing Fig. 4 it is evident that the effects of prebuckling in-plane deformation and curvature terms on the buckling factors are quite small in square plates. When the plate aspect ratio is greater than 1.25, the effect of prebuckling in-plane deformation can be of the same order of magnitude as the shear deformation effect (Fig. 4).

Figure 10 illustrates the influence of the number of plies in square plates ($t/b = 0.1$) subject to compression along the x-axis and tension along the y-axis. The variation in buckling factors between Case 1 and Case 2, 3 or 4 changes slightly as the number of plies increases.

Finally, the variation of buckling factor λ with the in-plane load ratios N_y/N_x for square plates is examined in Figs 11 and 12. When the plates are subjected to tension along the y-axis (Fig. 11), the inclusion of prebuckling in-plane deformation and curvature terms (Cases 2–4) decreases the buckling factors by comparison with Case 1. The decrease is significant for $N_y/N_x < -0.4$. The maximum difference in buckling factors between Cases

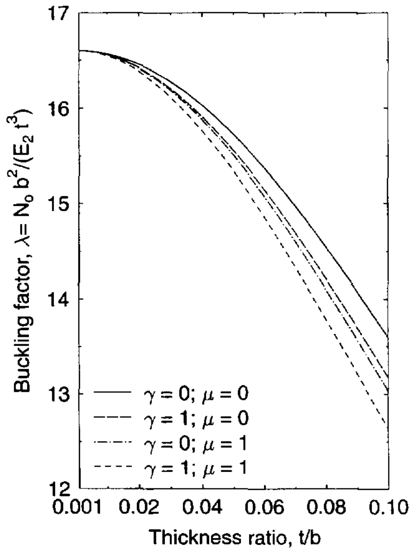

Fig. 9. Buckling factor, λ, vs thickness ratio, t/b, for simply supported symmetric cross-ply square laminates ($E_1/E_2 = 3$, $a/b = 1$, three-ply) subject to compression along the x-axis and tension along the y-axis ($N_x = N_o$, $N_y = -N_o$) with different combinations of the effects of prebuckling in-plane deformation and curvature terms.

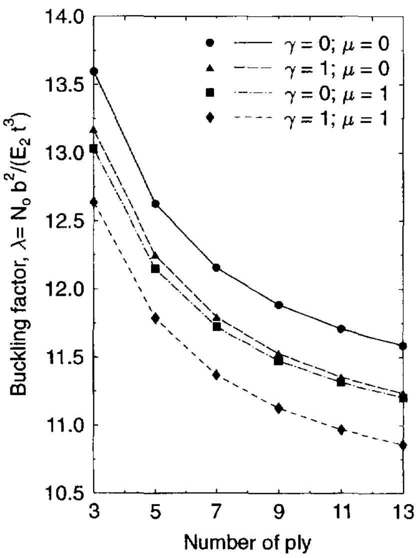

Fig. 10. Buckling factor, λ, vs number of plies, n, for simply supported symmetric cross-ply square laminates ($E_1/E_2 = 3$, $a/b = 1$, $t/b = 0.1$) subject to compression along the x-axis and tension along the y-axis ($N_x = N_o$, $N_y = -N_o$) with different combinations of the effects of prebuckling in-plane deformation and curvature terms.

1 and 4 is about 20% at $N_y/N_x \approx -0.55$. Mode shape shift occurs in Cases 1 and 3 at $N_y/N_x \approx -0.55$ and in Cases 2 and 4 at $N_y/N_x \approx -0.75$. For plates subject to biaxial compression (Fig. 12), the effects of prebuckling in-plane deformation and curvature terms (Case 2–4) on the buckling factors are insignificant.

6. CONCLUSIONS

Using the first-order shear deformation plate theory, the total incremental energy functional has been derived for symmetric cross-ply laminated rectangular plates subject to in-plane loads with consideration of prebuckling in-plane deformation and higher-order terms (curvature terms). The governing differential equations for the plate system have

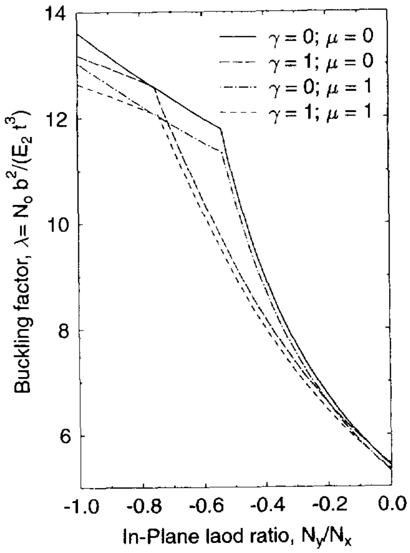

Fig. 11. Buckling factor, λ, vs in-plane load ratio, $\beta = N_y/N_x$, for simply supported symmetric cross-ply square laminates ($E_1/E_2 = 3$, $a/b = 1$, $t/b = 0.1$, $N_x = N_o$, $N_y \leqslant 0$) with different combinations of the effects of prebuckling in-plane deformation and curvature terms.

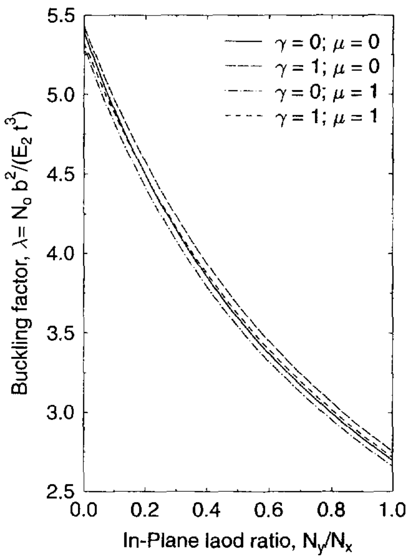

Fig. 12. Buckling factor, λ, vs in-plane load ratio, $\beta = N_y/N_x$, for simply supported symmetric cross-ply square laminates ($E_1/E_2 = 3$, $a/b = 1$, $t/b = 0.1$, $N_x = N_o$, $N_y \geqslant 0$) with different combinations of the effects of prebuckling in-plane deformation and curvature terms.

been obtained by applying stationary conditions to the derived energy functional. Using the Navier solution procedure, the characteristic eigenvalue equation has been derived for simply supported rectangular laminates.

In order to investigate the effects of prebuckling in-plane deformation and curvature terms on the buckling behaviour of plates, closed-form critical buckling solutions have been generated for plates with various loading conditions, aspect ratios, thickness ratios, degrees of orthotropy and numbers of plies. From this numerical study, the following conclusions may be drawn:

- The inclusion of prebuckling in-plane deformation may increase or decrease the buckling factor for plates subject to uniaxial compression, depending on the aspect ratio. It will always decrease the buckling factor for plates subject to compression along the x-axis and tension along the y-axis, but increase the buckling factor for plates under biaxial compression.

- The inclusion of curvature terms decreases the buckling factor no matter what loading condition is applied.
- The effects of prebuckling in-plane deformation and curvature terms on the buckling factor are less significant for plates subject to biaxial compression and more significant for plates subjected to compression along the x-axis and tension along the y-axis. These effects should not be neglected when plates are loaded by uniaxial compression or tension in one direction and compression in another as they may be of the same order of magnitude as the shear deformation effect.
- Increasing the degree of orthotropy slightly changes the effects of prebuckling in-plane deformation and curvature terms on the buckling factor.
- When a plate is thin ($t/b \leqslant 0.001$), the effects of prebuckling in-plane deformation and curvature terms on the buckling factors are negligible. These effects increase as the plate becomes thicker.
- The number of plies of a plate appears to have only a small influence on the effects of prebuckling in-plane deformation and curvature terms.

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